SPHERICAL NUMBERS: A N-DIMENSIONAL DIVISION ALGEBRA OVER THE REALS.

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Abstract

I present spherical numbers in the group $\mathbb{V}$ and show that the numbers $\mathbb{V}$ of dimension $\mathbb{R}^{n+1}$, $n \geqslant 1$, form a finite to infinite-dimensional division algebra over $\mathbb{R}$ that is associative, commutative and distributive. As an application, I derive a new multi-dimensional solution to the wave equation that describe spherical wave objects, whose centre propagate with velocity c and the object intensity does not dilute with time or distance.

A spherical-number $\mathbb{V}$ is a point on a sphere in a complex volume that is defined by orthogonal orientated imaginary unit numbers $i_1, i_2, \ldots, i_n$. The group $\mathbb{V}$ is a set with four operations; these are the two binary operators for spherical-multiplication and spherical-addition [1], and two unitary operators for spherical-exponentiation and spherical-negation. The group is characterised by:

(i) $\mathbb{V}$ is an abelian group with respect to spherical-multiplication with identity one,

(ii) $\mathbb{V}$ is an abelian group with respect to spherical-addition with identity zero,

(iii) the distributive laws hold.

If the term spherical is dropped, (i) and (ii) remain true and the distributive laws over cartesian-addition is lost. The innovative step to define these numbers, was not in defining a Hamilton like operation $ijk = -1$, but using the same rule that defines the spherical numbers to eliminate the product terms $i_g i_h$. The complex numbers $\mathbb{C}$ are a subset of $\mathbb{V}$.

1 DEFINING THE SPHERICAL NUMBERS.

The spherical number is expressed in the Euler form as

$$a = r\,e^{{}_s i\,{}_s\theta},$$

where e is the natural number, ${}_s i$ [2] is the spherical-imaginary number, ${}_s\theta$ a spherical angle. In the cartesian form the spherical number is expressed as

$$r\,e^{{}_s i\,{}_s\theta} = a_0 i_0 + a_1 i_1 + a_2 i_2 + \cdots + a_n i_n,$$

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[1] spherical-addition is not to be confused with cartesian- or vector-addition

[2] the preceding subscript s reads 'spherical'

where n defines the order of the $(n+1)$-dimensional spherical number, the coefficients $a_k \in \mathbb{R}$, i_0 is the multiplicative identity $1_{\mathbb{V}}$ and usually omitted in the expression, and $i_1, i_2, \ldots, i_n$ are imaginary unit-numbers orthogonally orientated to i_0, and to each other, and $r = \sqrt{a_0^2 + a_1^2 + a_2^2 + \cdots + a_n^2}$.

The values of the coefficients $a_0, a_1, a_2, \ldots, a_n$ are determined by the spherical angle

$$_s\theta = {}_s(\theta_1, \theta_2, \ldots, \theta_n)\,.$$

The angles $\theta_1, \theta_2, \ldots, \theta_n$ can be interpreted as a series of ordered successive rotations of the number (thought as a vector) ri_0 into the complex volume, starting with the rotation of ri_0 by θ_1 towards the i_1-axis and ending with the rotation of $r\, e^{_si\,_s(\theta_1,\theta_2,\ldots,\theta_{n-1})}$ by θ_n towards the i_n-axis.

Definition 1. *Products of imaginary unit-numbers*: The $\mathbb{V}$-algebra defines three rules (the fourth is derived):

$$_si^2 \Rightarrow -1 \tag{1}$$

$$i_h^2 \Rightarrow -e^{_si\,_s\left(\theta_1,\theta_2,\ldots,\theta_g\right)}, \qquad h = g+1 \tag{2}$$

$$i_h^3 \Rightarrow -i_h \tag{3}$$

$$i_h e^{_si\,_s\left(\theta_1,\theta_2,\ldots,\theta_g\right)} \Rightarrow i_h, \qquad h = g+1 \tag{4}$$

Using above rules, the Euler form of the spherical number can be expanded to a series of products

$$r\, e^{_si\,_s\theta} = r\, e^{i_1\theta_1}\, e^{i_2\theta_2}\, e^{i_3\theta_3} \cdots e^{i_n\theta_n} \tag{5}$$

and represents a point on a n-sphere of radius r in the complex volume.

Example 1. The three-dimensional spherical number

$$\begin{aligned} a &= e^{_si\,_s(\alpha_1,\alpha_2)} = e^{i_1\alpha_1}\, e^{i_2\alpha_2} \\ &= e^{i_1\alpha_1}(\cos\alpha_2 + i_2 \sin\alpha_2) \\ &= e^{i_1\alpha_1} \cos\alpha_2 + i_2 \sin\alpha_2 \qquad \text{after applying rule (4)} \\ &= \cos\alpha_1 \cos\alpha_2 + i_1 \sin\alpha_1 \cos\alpha_2 + i_2 \sin\alpha_2 \end{aligned}$$

Definition 2. *The binary operation of spherical-multiplication*, expressed with the symbol $\cdot$, is performed by addition of the rotation angles, and is defined as:

$$r_a e^{_si\,_s\alpha} \cdot r_b e^{_si\,_s\beta} = r_a r_b e^{_si\,_s(\alpha_1+\beta_1,\alpha_2+\beta_2,\ldots,\alpha_n+\beta_n)} \tag{6}$$

Definition 3. *The unitary operation of exponentiation* is performed by multiplying each rotation angle by the exponent x, and is defined as

$$\left(e^{_si\,_s(\theta_1,\theta_2,\ldots,\theta_n)}\right)^x = e^{_si\,_s(x\theta_1,x\theta_2,\ldots,x\theta_n)} \tag{7}$$

which includes inversion when $x = -1$, therefore

$$e^{_si\,_s\theta} \cdot \left(e^{_si\,_s\theta}\right)^{-1} = 1 = \left(e^{_si\,_s\theta}\right)^{-1} \cdot e^{_si\,_s\theta} \tag{8}$$

Definition 4. *The binary operation of spherical-addition*, is expressed with the symbol $\oplus$, is defined as:

$$r_a e^{s i \, s\alpha} \oplus r_b e^{s i \, s\beta} = r_s e^{s i \, s\sigma} \tag{9}$$

where

$$r_s = \sqrt{r_a^2 + r_b^2 + 2 r_a r_b \frac{1}{m} \sum_{g=1}^{m} \cos(\alpha_g - \beta_g)} \tag{10}$$

$$\sigma_h = \arctan\left(r_a \cos\alpha_h + r_b \cos\beta_h \;,\; r_a \sin\alpha_h + r_b \sin\beta_h\right) \tag{11}$$

and $1 \leqslant h \leqslant m$ and m is the largest value so that $\alpha_m \neq 0 \neq \beta_m$ holds, and $\arctan(x, y) = \arctan y/x$ after taking into account which quadrant the point (x, y) is in, and $\arctan(x, 0) = 0$.

Definition 5. *The unitary operation of negation* is defined by adding π to the highest order non zero rotation angle θ_m, defined as

$$e^{s i \, s(\theta_1, \theta_2, \ldots, \theta_m + \pi)} = -e^{s i \, s(\theta_1, \theta_2, \ldots, \theta_m)} \tag{12}$$

where m is the largest value so that $\theta_m \neq 0$ holds, therefore

$$\left(-e^{s i \, s\theta}\right) \oplus e^{s i \, s\theta} = 0 = e^{s i \, s\theta} \oplus \left(-e^{s i \, s\theta}\right) \tag{13}$$

Without further ado:

Theorem 1. Spherical numbers of any order n, i.e. of dimension $\mathbb{R}^{n+1}$, for all $n \geqslant 1$ populate the field $\mathbb{V}$ to form a finite to infinite-dimensional associative, commutative and distributive algebra over the reals without divisors of zero.

Proof. By using the above five definitions, and the trigonometric identity

$$\arctan\left(\frac{a\sin(\lambda + \alpha) + b\sin(\lambda + \beta)}{a\cos(\lambda + \alpha) + b\cos(\lambda + \beta)}\right) = \lambda + \arctan\left(\frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}\right) \tag{14}$$

the distributive property is confirmed, i.e.

$$\left(r_c e^{s i \, s\lambda} \cdot r_a e^{s i \, s\alpha}\right) \oplus \left(r_c e^{s i \, s\lambda} \cdot r_b e^{s i \, s\beta}\right) = r_c e^{s i \, s\lambda} \cdot \left(r_a e^{s i \, s\alpha} \oplus r_b e^{s i \, s\beta}\right) \tag{15}$$

□

Note 1. The first order spherical numbers, i.e. two-dimensional numbers, are the complex numbers and first order spherical-addition is equivalent to vector-addition over the cartesian coordinates.

2 Spherical numbers and cartesian representation.

We note that in a multi-dimensional construct that $\sqrt{-1}$ has the solution

$$\sqrt{-1} = i_1 \text{ or } -i_1.$$

This does not imply the definition for i_1 by $i_1^2 = -1$ but is the result of the exponentiation by a half of a number $e^{s i \, s(g\pi)}$, where $g = 2h - 1$ and $-\infty < h < \infty$.

If a spherical number is expressed in its cartesian form $a_0 i_0 + a_1 i_1 + \cdots + a_n i_n$ we note the cartesian-equivalences over π when the pair $(\theta_{2x-1}, \theta_{2x})$ is replaced with $(\pi \pm \theta_{2x-1}, \pi \mp \theta_{2x})$, i.e.

$$e^{s i \, s(\theta_1, \theta_2, \ldots, \theta_l, \theta_m, \theta_n)} = e^{s i \, s(\pi + \theta_1, \pi - \theta_2, \ldots, \pi - \theta_l, \pi + \theta_m, \theta_n)},$$

and we also note that if $x = a_n i_n$ then the Euler form is indeterminate as all information regarding the rotation angles $\theta_1, \theta_2, \ldots, \theta_{n-1}$ is lost.

Therefore avoid expressing spherical numbers the cartesian form, the only correct algebraic representation is in the Euler form.

Spherical numbers can be used in calculus. Any function $f(x)$, which is smooth in $\mathbb{R}$ is also smooth in $\mathbb{V}$. Differentiating and integrating

$$\varphi = e^{{}_si\ {}_s\alpha} = e^{{}_si\ {}_s(\alpha_1,\alpha_2,\ldots,\alpha_n)}$$

is performed with respect to the variable ${}_s\alpha$ and not its individual components. The accustomed rules apply: [3]

$$\frac{d\varphi}{d\,{}_s\alpha} = {}_si\ e^{{}_si\ {}_s\alpha} \qquad \text{and} \qquad \int \varphi\ d\,{}_s\alpha = -{}_si\ e^{{}_si\ {}_s\alpha} \tag{16}$$

If φ is a function of t,

$$\varphi(t) = e^{{}_si\ {}_s\omega t} = e^{{}_si\ {}_s(\omega_1,\omega_2,\ldots,\omega_n)t}$$

then

$$\frac{d\varphi(t)}{dt} = {}_si\,|{}_s\omega|\,e^{{}_si\ {}_s\omega t} \qquad \text{and} \qquad \int \varphi(t)\ dt = -{}_si\ \frac{e^{{}_si\ {}_s\omega t}}{|{}_s\omega|} \tag{17}$$

where $|{}_s\omega| = \sqrt{\omega_1^2 + \omega_2^2 + \cdots + \omega_n^2}$.

4 SPHERICAL WAVES

A multi-dimensional solution to the wave equation can be found, by using the *one-dimensional* form of the equation

$$c^2\frac{\partial^2\Psi}{\partial p^2} - \frac{\partial^2\Psi}{\partial t^2} = 0, \tag{18}$$

where ∂p is any partial increment in space; ∂t is a partial increment in time; and c is the propagation velocity defined by the characteristic of the space.

We begin with the assumptions that the n-dimensional function Ψ exists, and that Ψ is simultaneously a function of position $\mathcal{P}(p)$ and a function of time $\mathcal{T}(t)$. Consequently, a third simultaneous function can be formulated as the product of the square roots of the time and position functions

$$\Psi(p,t) = \begin{cases} \mathcal{P}(p) = \mathcal{X}^2(p) \\ \mathcal{T}(t) = \mathcal{Y}^2(t) \\ \mathcal{X}(p)\mathcal{Y}(t), \end{cases} \tag{19}$$

where $\mathcal{X}$ is a function of p which is a point or position in space, and $\mathcal{Y}$ a function of t. Any variations in $\mathcal{X}$ and $\mathcal{Y}$ can be independently looked at. Now, by taking the second partial derivatives of Ψ, once with respect to p and once with respect to t and introducing the results of these partial derivatives of $\Psi = \mathcal{X}(p)\mathcal{Y}(t)$ into (18), we find that

$$\frac{c^2}{\mathcal{X}(p)}\frac{d^2\mathcal{X}(p)}{dp^2} = \frac{1}{\mathcal{Y}(t)}\frac{d^2\mathcal{Y}(t)}{dt^2} \quad = -\frac{{}_s\omega^2}{4}$$

where the third term is introduced in anticipation of the desired result. The derivatives are total as $\mathcal{X}(p)$ and $\mathcal{Y}(t)$ are independent of one another, but also

[3] In fact, φ can be transformed by a series of axis or Euler rotations to a complex number $\varphi = e^{j\alpha'}$, where the new complex plane that defines $(1,j)$ is any plane in the complex volume $(1,i_1,i_2,\ldots,i_n)$.

equal to each other if and only if $p = ct$. The solutions of the two second order differential equations are

$$\mathcal{X}(p) = \sqrt{\mathcal{A}} e^{{}_s i \, {}_s\omega p/2c}$$
$$\mathcal{Y}(t) = \sqrt{\mathcal{A}} e^{{}_s i \, {}_s\omega t/2}$$

where $\mathcal{A}$ is an arbitrary constant, independent of time or position and characterises the function Ψ. All that remains is to square the above functions, to write the solution of (18) by introducing the results into (19)

$$\Psi(p,t) = \begin{cases} \mathcal{P}(p) = \mathcal{A} e^{{}_s i \, {}_s\omega p/c} \\ \mathcal{T}(t) = \mathcal{A} e^{{}_s i \, {}_s\omega t} \end{cases} , \quad p = ct. \tag{20}$$

The function $\Psi(p,t)$ now has multi-dimensional properties. The first order solution could describe a propeller like function propagating along its axis of rotation, its tips tracing the path of a helix, as depicted in Figure 1. The order of the solution can be increased to describe spherical wave objects, whose centre propagate with velocity c and the object intensity does not dilute with time or distance.

This solution differs from the plain-wave solution normally derived to describe, say, the spherical expansion of a compression/rarefaction in a media.

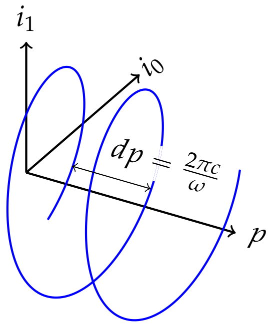

Figure 1:

5 SPHERICAL EIGENVALUES

Let's consider $\Psi(t)$ the time component of the wave function (20)

$$\Psi(t) = e^{{}_s i \, {}_s\omega t}$$

It is a spherical object. As per its definition it also is an eigenfunction of the linear operator $\Lambda_\Psi = \frac{d}{dt^2}$ such that

$$\Lambda_\Psi \Psi(t) = -\lambda \Psi(t) \tag{21}$$

where $\lambda = {}_s\omega^2$. Eigenvalues are usually associated with boundary conditions i.e. the two end points of a vibrating string. A boundary condition for $\Psi(t)$ exists only if the spatial angle

$${}_s\omega t = {}_s(\omega_1, \omega_2, \omega_3 \ldots \omega_n)\, t \tag{22}$$

is so dimensioned that it is ensured that the path followed from any point on the spherical curve, defined by $\Psi(t)$, always repeats itself exactly in time. Therefore any point on the spherical curve defined by $\Psi(t)$ can be used as a boundary. This only happens when the ratios

$$\frac{\omega_1 : \omega_2 : \ldots : \omega_n}{\min(\omega_1, \omega_2, \ldots \omega_n)} \tag{23}$$

are integer ratios.

Not only do the numbers $\mathbb{V}$ introduce the new aspect into multi-dimensional algebra/calculus, I can also imagine that this algebra may be helpful in formulating new descriptions for the physical world. Even if, for whatever reason, the counter-intuitive spherical-addition is omitted [4] an abelian group remains, as outlined in points (i) and (ii) of the introductory paragraph.

Technically, the Frobenius 1877 theorem on division algebra, and many related theorems, are now falsified. I thus conclude with the philosophical observation:

THE TRUTH IS: If a theorem is interpreted [5] as "*a general proposition not self-evident but proved by a chain of reasoning; a truth established by means of accepted truths*" then there exist theorems that will be falsified in the future, if and only if our current knowledge of accepted truths is subordinated to a future superordinate knowledge of accepted truths.

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[4] The spherical-addition would be omitted if the algebraic vector product is introduced into vector algebra.
[5] as defined in Oxford dictionary